COXETER SYSTEM OF LINES ARE SETS OF INJECTIVITY FOR THE TWISTED SPHERICAL MEANS ON $\mathbb{C}$

RAJESH K. SRIVASTAVA

Dedicated to Prof. E. M. Stein on the occasion of his 80th birthday.

ABSTRACT. It is well known that a line in $\mathbb{R}^2$ is not a set of injectivity for the spherical means for odd functions about that line. We prove that any line passing through the origin is a set of injectivity for the twisted spherical means (TSM) for functions $f \in L^2(\mathbb{C})$, whose each of spectral projection $e^{\frac{1}{4}|z|^2} f \times \varphi_k$ is a polynomial. Then, we prove that any Coxeter system of even number of lines is a set of injectivity for the TSM for $L^q(\mathbb{C})$, $1 \leq q \leq 2$.

1. Introduction

In an interesting result, Courant and Hilbert ([8], p. 699) had proved that if average of an even function about a line L vanishes over all circles centered at L, then $f \equiv 0$. As a consequence of this result, any line L in $\mathbb{R}^2$ is not a set of injectivity for the spherical means for the odd functions about L.

However, this result does not continue to hold for injectivity of the twisted spherical means on complex plane $\mathbb{C}$, because of non-commutative nature of underlying geometry of the Heisenberg group, (see [5, 6, 7, 11]). The question, in general that any real analytic curve can be a set of injectivity for the twisted spherical mean for $L^2(\mathbb{C})$, is still an open problem. However, we are able to prove the following partial results for the TSM.

Let $f \in L^2(\mathbb{C})$ be such that for each $k \in \mathbb{Z}_+$ (set of non-negative integers), the projection $e^{\frac{1}{4}|z|^2} f \times \varphi_k$ is a polynomial. Suppose the twisted spherical mean $f \times \mu_r(x)$ of the function f vanishes $\forall r > 0$ and $\forall x \in \mathbb{R}$. Then $f = 0$ a.e. That is, the X-axis is a set of injectivity for the TSM on $\mathbb{C}$.

By rotation, it follows that any line passing through the origin is a set of injectivity for the TSM. Since $f \times \varphi_k$ is a real analytic function, in the above case, we only need the centers to be a sequence in $\mathbb{R}$ having a limit point.

With the same exponential condition, we observe that any curve $\gamma(t) = (\gamma_1(t), \gamma_2(t))$, which passes through the origin, where $\gamma_j,\ j = 1, 2$ are polynomials is also a set of injectivity for the TSM. It is an interesting question, wether a real analytic curve is a set of injectivity for the TSM for a smooth class of functions.

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Further, to complete the arguments of our idea, we prove that any two perpendicular lines is a set of injectivity for the TSM on $L^q(\mathbb{C})$. Moreover, this result implies that any Coxeter system of even number of lines is a set of injectivity for the TSM on $L^q(\mathbb{C})$. These results for the twisted spherical means are in sharp contrast to the well known result for injectivity of the Euclidean spherical means on $\mathbb{R}^2$, due to Agranovsky and Quinto [2].

On the basis of these striking results, it is therefore natural to ask, whether any Coxeter system of odd number of lines can be a set of injectivity for the TSM. We believe, our techniques with slight modifications would continue to work for any Coxeter system of lines to be a set of injectivity for the TSM, which we leave open for the time being.

In 1996, Agranovsky and Quinto have proved a major breakthrough result in the *integral geometry,* which completely characterizes the sets of injectivity for the spherical means on the space of compactly supported continuous functions on $\mathbb{R}^2$. Their result says that the exceptional set for the sets of injectivity is a very thin set which consists of a Coxeter system of lines union finitely many points.

Let μ_r be the normalized surface measure on sphere $S_r(x)$. Let $\mathscr{F} \subseteq L^1_{\text{loc}}(\mathbb{R}^n)$. We say that $S \subseteq \mathbb{R}^n$ is a set of injectivity for the spherical means for $\mathscr{F}$ if for $f \in \mathscr{F}$ with $f * \mu_r(x) = 0, \forall r > 0$ and $\forall x \in S$, implies $f = 0$ a.e.

Theorem 1.1. [2] *A set $S \subset \mathbb{R}^2$ is a set of injectivity for the spherical means for $C_c(\mathbb{R}^2)$ if and only if $S \nsubseteq \omega(\Sigma_N) \cup F$, where ω is a rigid motion of $\mathbb{R}^2$, $\Sigma_N = \cup_{l=0}^{N-1}\{te^{\frac{i\pi l}{N}} : t \in \mathbb{R}\}$ is a Coxeter system of N lines and F is a finite set in $\mathbb{R}^2$.*

In particular, any closed curve is a set of injectivity for $C_c(\mathbb{R}^2)$. In fact, Agranovsky et al. [1] further prove that the boundary of any bounded domain in $\mathbb{R}^n$ $(n \geq 2)$ is set of injectivity for the spherical means on $L^p(\mathbb{R}^n)$, with $1 \leq p \leq \frac{2n}{n-1}$. For $p > \frac{2n}{n-1}$, unit sphere S^{n-1} is an example of non-injectivity set in $\mathbb{R}^n$. This result has been generalized for certain weighted spherical means, see [15]. In general, the question of set of injectivity for the spherical means with real analytic weight is still open. In [15], it has been shown that S^{n-1} is a set of injectivity for the spherical means with real analytic weights for the class of radial functions.

An analogue of Theorem 1.1 in the higher dimensions is still open and appeared as a conjecture in their work [2]. It says that the sets of non-injectivity for the Euclidean spherical means are contained in a certain algebraic variety. Following is their conjecture.

Conjecture [2]**.** A set $S \subset \mathbb{R}^n$ is a set of injectivity for the spherical means for $C_c(\mathbb{R}^2)$ if and only if $S \nsubseteq \omega(\Sigma) \cup F$, where ω is a rigid motion of $\mathbb{R}^n$, Σ is the zero set of a homogeneous harmonic polynomial and F is an algebraic variety in $\mathbb{R}^n$ of co-dimension at most 2.

This conjecture remains unsolved, however a partial result related to this conjecture has been proved by Kuchment et al. [4]. They also present a brief survey on the recent development towards the above conjecture. However, in

this article, we observe that this conjecture does not continue to hold for the spherical means on the Heisenberg group $\mathbb{H}^1 = \mathbb{C} \times \mathbb{R}$. In fact result on $\mathbb{H}^1$ is an adverse to the Euclidean result, Theorem 1.1 on $\mathbb{R}^2$.

In more general, let $f \in L^1_{\text{loc}}(\mathbb{C}^n)$ and write $S(f) = \{z \in \mathbb{C}^n : f \times \mu_r(z) = 0, \forall r > 0\}$. Our main problem is to describe completely the geometrical structure of $S(f)$ that would characterize which "sets" are set of injectivity for the TSM. For example, let f be a type function $f(z) = \tilde{a}(|z|)P(z) \in L^2(\mathbb{C}^n)$, where $P \in H_{p,q}$. Here $H_{p,q}$ is the space of homogeneous harmonic polynomials on $\mathbb{C}^n$ of type

$$P(z) = \sum_{|\alpha|=p, |\beta|=q} C_{\alpha\beta} z^\alpha \bar{z}^\beta.$$

Then $S(f) \subseteq P^{-1}(0)$. This means a set $S \subset \mathbb{C}^n$ is set of injectivity for twisted spherical means for type functions if and only if $S \nsubseteq P^{-1}(0)$. Since P is harmonic, by maximal principle $P^{-1}(0)$ can not contain the boundary of any bounded domain in $\mathbb{C}^n$. Hence the boundary of any bounded domain would be a possible candidate for set of injectivity for the TSM. The question that the boundary of the bounded domain is a set of injectivity for the TSM has been taken up by many authors (see [3, 16, 21]). In a result of Narayanan and Thangavelu [16], it has been proved that the spheres centered at the origin are sets of injectivity for the TSM on $\mathbb{C}^n$. The author has generalized the result of [16] for certain weighted twisted spherical means, (see [20]). In general, the question of set of injectivity for the twisted spherical means (TSM) with real analytic weight is still open.

2. Notation and Preliminaries

We define the twisted spherical means which arise in the study of spherical means on Heisenberg group. The group $\mathbb{H}^n$, as a manifold, is $\mathbb{C}^n \times \mathbb{R}$ with the group law

$$(z,t)(w,s) = (z + w, t + s + \frac{1}{2}\text{Im}(z.\bar{w})), \; z, w \in \mathbb{C}^n \text{ and } t, s \in \mathbb{R}.$$

Let μ_s be the normalized surface measure on the sphere $\{(z, 0) : |z| = s\} \subset \mathbb{H}^n$. The spherical means of a function f in $L^1(\mathbb{H}^n)$ are defined by

$$f * \mu_s(z,t) = \int_{|w|=s} f((z,t)(-w,0)) \, d\mu_s(w). \tag{2.1}$$

Thus the spherical means can be thought of as convolution operators. An important technique in many problem on $\mathbb{H}^n$ is to take partial Fourier transform in the t-variable to reduce matters to $\mathbb{C}^n$. Let

$$f^\lambda(z) = \int_{\mathbb{R}} f(z,t) e^{i\lambda t} dt$$

be the inverse Fourier transform of f in the t-variable. Then a simple calculation shows that

$$\begin{aligned}(f*\mu_s)^\lambda(z) &= \int_{-\infty}^{\infty} f*\mu_s(z,t)e^{i\lambda t}dt \\ &= \int_{|w|=s} f^\lambda(z-w)e^{\frac{i\lambda}{2}\mathrm{Im}(z.\bar{w})}\, d\mu_s(w) \\ &= f^\lambda \times_\lambda \mu_s(z),\end{aligned}$$

where μ_s is now being thought of as normalized surface measure on the sphere $S_s(o)=\{z\in\mathbb{C}^n : |z|=s\}$ in $\mathbb{C}^n$. Thus the spherical mean $f*\mu_s$ on the Heisenberg group can be studied using the λ-twisted spherical mean $f^\lambda \times_\lambda \mu_s$ on $\mathbb{C}^n$. For $\lambda\neq 0$, a further scaling argument shows that it is enough to study these means for the case of $\lambda=1$.

Let $\mathscr{F}\subseteq L^1_{\text{loc}}(\mathbb{C}^n)$. We say $S\subseteq\mathbb{C}^n$ is a set of injectivity for twisted spherical means for $\mathscr{F}$ if for $f\in\mathscr{F}$ with $f\times\mu_r(z)=0, \forall r>0$ and $\forall z\in S$, implies $f=0$ a.e. The results on set of injectivity differ in the choice of sets and the class of functions considered. We would like to refer to [3, 16, 20], for some results on the sets of injectivity for the TSM.

We need the following basic facts from the theory of bigraded spherical harmonics (see [24], p.62 for details). We shall use the notation $K=U(n)$ and $M=U(n-1)$. Then, $S^{2n-1}\cong K/M$ under the map $kM\to k.e_n$, $k\in U(n)$ and $e_n=(0,\ldots,1)\in\mathbb{C}^n$. Let $\hat{K}_M$ denote the set of all equivalence classes of irreducible unitary representations of K which have a nonzero M-fixed vector. It is known that each representation in $\hat{K}_M$ has a unique nonzero M-fixed vector, up to a scalar multiple.

For a $\delta\in\hat{K}_M$, which is realized on V_δ, let $\{e_1,\ldots,e_{d(\delta)}\}$ be an orthonormal basis of V_δ with e_1 as the M-fixed vector. Let $t^\delta_{ij}(k)=\langle e_i,\delta(k)e_j\rangle$, $k\in K$ and $\langle,\rangle$ stand for the innerproduct on V_δ. By Peter-Weyl theorem, it follows that $\{\sqrt{d(\delta)}t^\delta_{j1} : 1\leq j\leq d(\delta), \delta\in\hat{K}_M\}$ is an orthonormal basis of $L^2(K/M)$ (see [24], p.14 for details). Define $Y^\delta_j(\omega)=\sqrt{d(\delta)}t^\delta_{j1}(k)$, where $\omega=k.e_n\in S^{2n-1}$, $k\in K$. It then follows that $\{Y^\delta_j : 1\leq j\leq d(\delta), \delta\in\hat{K}_M,\}$ forms an orthonormal basis for $L^2(S^{2n-1})$.

For our purpose, we need a concrete realization of the representations in $\hat{K}_M$, which can be done in the following way. See [19], p.253, for details. For $p,q\in\mathbb{Z}_+$, let $P_{p,q}$ denote the space of all polynomials P in z and $\bar{z}$ of the form

$$P(z)=\sum_{|\alpha|=p}\sum_{|\beta|=q} c_{\alpha\beta}z^\alpha\bar{z}^\beta.$$

Let $H_{p,q}=\{P\in P_{p,q} : \Delta P=0\}$. The elements of $H_{p,q}$ are called the bigraded solid harmonics on $\mathbb{C}^n$. The group K acts on $H_{p,q}$ in a natural way. It is easy to see that the space $H_{p,q}$ is K-invariant. Let $\pi_{p,q}$ denote the corresponding representation of K on $H_{p,q}$. Then representations in $\hat{K}_M$ can be identified, up to unitary equivalence, with the collection $\{\pi_{p,q} : p,q\in\mathbb{Z}_+\}$.

Define the bigraded spherical harmonic by $Y_j^{p,q}(\omega) = \sqrt{d(p,q)}\,t_{j1}^{p,q}(k)$. Then $\{Y_j^{p,q} : 1 \leq j \leq d(p,q), p,q \in \mathbb{Z}_+\}$ forms an orthonormal basis for $L^2(S^{2n-1})$. Therefore, for a continuous function f on $\mathbb{C}^n$, writing $z = \rho\omega$, where $\rho > 0$ and $\omega \in S^{2n-1}$, we can expand the function f in terms of spherical harmonics as

$$f(\rho\omega) = \sum_{p,q\geq 0} \sum_{j=1}^{d(p,q)} a_j^{p,q}(\rho) Y_j^{p,q}(\omega), \tag{2.2}$$

where the series on the right-hand side converges uniformly on every compact set $K \subseteq \mathbb{C}^n$. The functions $a_j^{p,q}$ are called the spherical harmonic coefficients of f and function $a^{p,q}(\rho)Y^{p,q}(\omega)$ is known as the type function.

We need the Hecke-Bochner identity for the spectral projections $f \times \varphi_k^{n-1}$, for function $f \in L^2(\mathbb{C}^n)$. See [24], p.70. For $k \in \mathbb{Z}_+$, the Laguerre functions φ_k^{n-1} is defined by $\varphi_k^{n-1}(z) = L_k^{n-1}(\frac{1}{2}|z|^2)e^{-\frac{1}{4}|z|^2}$, where

$$L_k^{n-1}(x) = \sum_{j=0}^{k} (-1)^j \binom{k+n-1}{k-j} \frac{x^j}{j!},$$

is the Laguerre polynomial of degree k and order $n-1$.

Lemma 2.1. *Let $\tilde{a}P \in L^2(\mathbb{C}^n)$, where $\tilde{a}$ is a radial function and $P \in H_{p,q}$. Then*

$$\begin{aligned} \tilde{a}P \times \varphi_j^{n-1}(z) &= (2\pi)^n \left\langle \tilde{a}, \varphi_{k-p}^{n+p+q-1} \right\rangle P(z)\varphi_{k-p}^{n+p+q-1}(z) \\ &= (2\pi)^{-n} P(z)\, \tilde{a} \times \varphi_{k-p}^{n+p+q-1}(z), \end{aligned} \tag{2.3}$$

if $k \geq p$ and 0 otherwise. The convolution in the right hand side is on the space $\mathbb{C}^{n+p+q}$.

3. Sets of injectivity for the twisted spherical means

In this section, we prove that the X-axis is a set of injectivity for the TSM for a certain class of functions in $L^2(\mathbb{C})$. Then, we replicate the method to prove that X-axis together with Y-axis is a set of injectivity for the TSM for $L^q(\mathbb{C})$. In the later case, we deduce a density result for $L^p(\mathbb{C}),\ 2 \leq p < \infty$.

Since Laguerre function φ_k^{n-1} is an eigenfunction of the special Hermite operator $A = -\Delta_z + \frac{1}{4}|z|^2$, with eigenvalue $2k+n$, the projection $f \times \varphi_k^{n-1}$ is also an eigenfunction of A with eigenvalue $2k+n$. As A is an elliptic operator and eigenfunction of an elliptic operator is real analytic [14], the projection $f \times \varphi_k^{n-1}$ must be a real analytic function on $\mathbb{C}^n$. The real analyticity of $f \times \varphi_k^{n-1}$ can also be understood by the fact that $f \times \varphi_k^{n-1}$ can be extended to a holomorphic function on $\mathbb{C}^{2n}$. Therefore, any determining set for the real analytic functions is a set of injectivity for the TSM on $L^q(\mathbb{C}^n)$ with $1 \leq q \leq \infty$. For details on determining sets for real analytic functions, see [17, 18]. Next, we find an expansion for $f \times \varphi_k^0$ with help of Hecke-Bochner identities for spectral projection.

Proposition 3.1. *Let $f \in L^2(\mathbb{C})$. Then the real analytic expansion of $Q_k(z) = f \times \varphi_k^0(z)$ can be written as*

$$Q_k(z) = \sum_{p=0}^{k} C_{k-p}^{p0} z^p \varphi_{k-p}^p(z) + \sum_{q=0}^{\infty} C_k^{0q} \bar{z}^q \varphi_k^q(z). \tag{3.1}$$

Proof. We know that

$$f(z) = \sum_{p=0}^{\infty} a^{p0}(|z|) z^p + \sum_{q=0}^{\infty} a^{0q}(|z|) \bar{z}^q, \text{ for } z \in \mathbb{C}.$$

Since $f \in L^2(\mathbb{C})$, using the Hecke-Bochner identity for the spectral projections as in Lemma 2.1, we can express $f \times \varphi_k^0(z)$ as

$$Q_k(z) = f \times \varphi_k^0(z) = \sum_{p=0}^{k} C_{k-p}^{p0} z^p \varphi_{k-p}^p(z) + \sum_{q=0}^{\infty} C_k^{0q} \bar{z}^q \varphi_k^q(z),$$

where the series on the right-hand side converges to Q_k in $L^2(\mathbb{C})$. In order to show that the series converges uniformly on every compact set $K \subseteq \mathbb{C}$, it is enough to show that the series

$$h(z) = \sum_{q=k+1}^{\infty} C_k^{0q} \bar{z}^q \varphi_k^q(z),$$

converges uniformly on every ball $B_R(o)$ in $\mathbb{C}$. Since $Q_k \in L^2(\mathbb{C})$, it follows that $h \in L^2(\mathbb{C})$ and

$$\|h\|_{L^2(\mathbb{C})}^2 = \sum_{q=k+1}^{\infty} |C_k^{0q}|^2 \|\bar{z}^q \varphi_k^q\|_{L^2(\mathbb{C})}^2 < \infty.$$

Since

$$\begin{aligned} \|\bar{z}^q \varphi_k^q\|_{L^2(\mathbb{C})}^2 &= \int_0^\infty \int_{S^1} |r\omega|^{2q} \left(\varphi_k^q\right)^2 r dr d\omega \\ &= 2\pi \int_0^\infty \left(\varphi_k^{q+1-1}\right)^2 r^{2(q+1)-1} dr \\ &= 2\pi \; 2^q \frac{(k+q)!}{k!}. \end{aligned} \tag{3.2}$$

Therefore, the coefficients C_k^{0q}'s must satisfy an estimate of type

$$|C_k^{0q}| \leq C \left(\frac{k!}{2^{q+1}(k+q)!} \right)^{\frac{1}{2}}, \tag{3.3}$$

where C is a constant and independent of q. Now, let $|z| \leq R$. Then, we have

$$
\begin{aligned}
|h(z)| &\leq e^{-\frac{1}{4}|z|^2} \sum_{q=k+1}^{\infty} |C_k^{0q}||z|^q \left| \sum_{j=0}^{k} (-1)^j \binom{q+k}{k-j} \frac{(\frac{1}{2}|z|^2)^j}{j!} \right| \\
&\leq Ce^{-\frac{1}{4}|z|^2} \sum_{q=k+1}^{\infty} \left(\frac{k!}{2^{q+1}(k+q)!} \right)^{\frac{1}{2}} |z|^q \frac{(q+k)!}{k!q!} \sum_{j=0}^{k} \frac{(\frac{1}{2}|z|^2)^j}{j!} \\
&\leq Ce^{-\frac{1}{4}|z|^2} \sum_{q=k+1}^{\infty} \left(\frac{(q+k)!}{2^{q+1}k!q!} \right)^{\frac{1}{2}} \frac{|z|^q}{(q!)^{\frac{1}{2}}} e^{\frac{1}{2}|z|^2} \\
&\leq Ce^{\frac{1}{4}R^2} \sum_{q=k+1}^{\infty} \left(\frac{(q+k)!}{2^{q+1}k!q!} \right)^{\frac{1}{2}} \frac{R^q}{(q!)^{\frac{1}{2}}} < \infty.
\end{aligned}
$$

Thus the function h is real analytic on $\mathbb{C}$. That is, the right-hand side of (3.1) is a real analytic function which agreeing to the real analytic function Q_k a.e. on $\mathbb{C}$. Hence (3.1) is a real analytic expansion of Q_k. □

We would like to call (3.1) the *Hecke-Bochner-Laguerre series* for the spectral projections. We study this series carefully and use it to prove the most striking results, Theorems (3.4, 3.6) of this article.

Theorem 3.2. *Let $f \in L^2(\mathbb{C})$. Suppose $f \times \mu_r(x) = 0, \forall\ r > 0$ and $\forall\ x \in \mathbb{R}$. Then $f \times \varphi_0^0 \equiv f \times \varphi_1^0 \equiv 0$ on $\mathbb{C}$.*

Proof. Since $f \times \mu_r(x) = 0,\ \forall r > 0$ and $\forall x \in \mathbb{R}$, by polar decomposition, it follows that $Q_k(x) = f \times \varphi_k^0(x) = 0,\ \forall x \in \mathbb{R}$ and $\forall k \in \mathbb{Z}_+$. For $k = 0$, we have

$Q_0(x) = C_0^{00}\varphi_0^0(x) + C_0^{00}\varphi_0^0(x) + C_0^{01}x\varphi_0^1(x) + C_0^{02}x^2\varphi_0^2(x) + \cdots = 0,\ \forall\ x \in \mathbb{R}.$

On equating the coefficients of $1, x, x^2, \ldots$ to zero, we get $C_0^{0q} = 0, \forall q \geq 0$. Hence $Q_0 \equiv 0$ on $\mathbb{C}$. From Equation (3.1), we have

$$Q_1(x) = C_1^{00}\varphi_1^0(x) + C_0^{10}x\varphi_0^1(x) + \sum_{q=0}^{\infty} C_1^{0q}x^q\varphi_1^q(x) = 0,\ \forall\ x \in \mathbb{R}.$$

Using the argument $x \to -x$, it follows that

$$2C_1^{00}\varphi_1^0(x) + \sum_{m=0}^{\infty} C_1^{0,2m}x^{2m}\varphi_1^{2m}(x) = 0.$$

By equating coefficient of $1, x^2, x^4, \ldots,$ we get $C_1^{0,2m} = 0$, for $m = 0, 1, 2, \ldots.$ Hence the series of $Q_1(x)$ reduces to

$$Q_1(x) = C_0^{1,0}x\varphi_0^1(x) + \sum_{m=0}^{\infty} C_1^{0,2m+1}x^{2m+1}\varphi_1^{2m+1}(x) = 0.$$

By canceling $e^{-\frac{1}{4}x^2}$ in the above series, we have

$$C_0^{1,0}x + \sum_{m=0}^{\infty} C_1^{0,2m+1}x^{2m+1}\left(2m + 2 - \frac{1}{2}x^2\right) = 0.$$

On equation the coefficients of $x, x^3, x^5, \dots$, we get the following recursion relations

$$C_0^{1,0} = -2C_1^{0,1} \text{ and } C_1^{0,2m+1} = \frac{C_1^{0,1}}{2^{2m}(m+1)!}; \text{ for } m = 1, 2, 3, \dots. \tag{3.4}$$

Now, we can write $Q_1(z) = C_0^{1,0} z\varphi_0^1(z) + h(z)$, where the series

$$h(z) = \sum_{m=0}^{\infty} C_1^{0,2m+1} \bar{z}^{2m+1} \varphi_1^{2m+1}(z)$$

converges in $L^2(\mathbb{C})$. We claim that all the coefficients $C_1^{0,2m+1};\ m = 0, 1, 2, \dots$ are zero. On contrary suppose infinitely many of these coefficients are non-zero. Then, by the estimate (3.2) and the recursion relations (3.4), we have

$$\begin{aligned}
\|h\|^2_{L^2(\mathbb{C})} &= \sum_{m=0}^{\infty} \left|C_1^{0,2m+1}\right|^2 \left\|\bar{z}^{2m+1}\varphi_1^{2m+1}\right\|^2_{L^2(\mathbb{C})} \\
&= 2\pi \left|C_1^{0,1}\right|^2 \sum_{m=0}^{\infty} \frac{2^{2m+1}(2m+2)!}{\left(2^{2m}(m+1)!\right)^2} \\
&= 4\pi \left|C_1^{0,1}\right|^2 \sum_{m=0}^{\infty} \frac{(2m+2)!}{2^{2m}\left((m+1)!\right)^2} = \sum_{m=0}^{\infty} b_m = \infty.
\end{aligned}$$

The series on the right-hand side diverges by Raabe's test. (See, [13], p.36). Since

$$\lim_{m\to\infty} \left\{ m\left(\frac{b_m}{b_{m+1}} - 1\right)\right\} = -\frac{1}{2} < 1.$$

This contradicts the fact that the series h is $L^2(\mathbb{C})$ summable. Thus, we get $C_1^{0,2m+1} = 0$, for $m = 0, 1, 2, \dots$. Hence, we conclude that $Q_1 \equiv 0$. $\square$

Remark 3.3. Under the same assumptions as in Theorem 3.2, it would be interesting to know, whether $Q_k \equiv 0$ for $k \geq 2$. The argument used to show $Q_1 \equiv 0$ does not work in this case. In another attempt, using the recursion relations $L_k^n = L_k^{n-1} + \cdots + L_0^{n-1}$, $L_k^n - L_{k-1}^n = L_k^{n-1}$ and the result that $f \times \varphi_0^0 = f \times \varphi_1^0 = 0$, we can easily deduce that $f \times \varphi_2^0 = f \times \varphi_2^1$. But, we are not able to conclude any thing more on account of the facts that $f \times \varphi_2^0$ is an eigenfunction of the special Hermite operator A and $\varphi_2^1 = \varphi_2^0 + \varphi_1^0 + \varphi_0^0$.

However, we prove the following partial result that any line passing through the origin is a set of injectivity for the TSM for a certain class of functions in $L^2(\mathbb{C})$. Since for any $\sigma \in U(n)$, we have $f \times \mu_r(\sigma.z) = (\pi(\sigma)f) \times \mu_r(z)$. It follows that a set $S \subset \mathbb{C}$ is a set of injectivity for the TSM if and only if for each $\sigma \in U(n)$, the set $\sigma.S$ is a set of injectivity for the TSM. In view of this, it is enough to prove that the X-axis is a set of injectivity for the TSM.

Theorem 3.4. *Let $f \in L^2(\mathbb{C})$ and for each $k \in \mathbb{Z}_+$ the projection $e^{\frac{1}{4}|z|^2} f \times \varphi_k^0$ is a polynomial. Suppose $f \times \mu_r(x) = 0, \forall\ r > 0$ and $\forall\ x \in \mathbb{R}$. Then $f = 0$ a.e.*

Proof. Since $f \in L^2(\mathbb{C})$, by polar decomposition, the condition $f \times \mu_r(x) = 0, \forall\ r > 0$ and $\forall\ x \in \mathbb{R}$ is equivalent to $f \times \varphi_k^0(x) = 0, \forall\ k \geq 0$ and $\forall\ x \in \mathbb{R}$. From Equation (3.1), we have

$$f \times \varphi_k^0(z) = \sum_{p=0}^{k} C_{k-p}^{p0} z^p \varphi_{k-p}^p(z) + \sum_{q=0}^{\infty} C_k^{0q} \bar{z}^q \varphi_k^q(z).$$

By the given exponential condition, we can write $f \times \varphi_k^0(z) = P(z,\bar{z})e^{-\frac{1}{4}|z|^2}$. Let $z = te^{i\theta}$. Then for each fixed t, the function $f \times \varphi_k^0(te^{i\theta})$ is a trigonometric polynomial. Using the orthogonality of $e^{in\theta}$ it follows that there exist $m = m(k) \in \mathbb{Z}_+$ such that

$$f \times \varphi_k^0(z) = \sum_{p=0}^{k} C_{k-p}^{p0} z^p \varphi_{k-p}^p(z) + \sum_{q=0}^{m} C_k^{0q} \bar{z}^q \varphi_k^q(z). \tag{3.5}$$

Therefore, for each $k \in \mathbb{Z}_+$, we have

$$\sum_{p=0}^{k} C_{k-p}^{p0} x^p \varphi_{k-p}^p(x) + \sum_{q=0}^{m} C_k^{0q} x^q \varphi_k^q(x) = 0, \forall\ x \in \mathbb{R}.$$

The constant term in the above expansion $2C_k^{00} = 0$, hence

$$\sum_{p=1}^{k} C_{k-p}^{p0} x^p \varphi_{k-p}^p(x) + \sum_{q=1}^{m} C_k^{0q} x^q \varphi_k^q(x) = 0, \forall\ x \in \mathbb{R}.$$

On equating the coefficient of the highest degree term x^{m+2k} to zero, we get $C_k^{0m} = 0$. Similarly, continuing this argument up to x^{2k}, we obtained $C_k^{0m} = C_k^{0(m-1)} = \cdots = C_k^{01} = 0$. Then equate the coefficients of $x, x^2, \ldots, x^{2k-1}$ to zero, we find $C_{k-1}^{10} = C_{k-2}^{20} = \cdots = C_0^{p0} = 0$. Thus $f \times \varphi_k^0 \equiv 0, \forall k \geq 0$. Hence $f = 0$ a.e. This completes the proof. □

Remark 3.5. (a). Since $f \times \varphi_k$ is real analytic and the zero set of a real analytic function is isolated, in this case, we only need the centres to be a sequence in $\mathbb{R}$ having a limit point. It is clear from (3.5) that any curve $\gamma := \{(\gamma_1(t), \gamma_2(t)) : t \in \mathbb{R}\}$, where $\gamma_j,\ j = 1, 2$ are polynomials is also a set of injectivity for the TSM. A natural question is that $\gamma := (\gamma_1, \gamma_2)$ with γ_j's are real analytic is a set of injectivity for the TSM. We believe that this will help in characterizing non-injectivity sets for TSM.

(b). Let us consider the functions

$$f_m(z) = \sum_{p=0}^{\infty} a^{p0}(|z|)z^p + \sum_{q=0}^{m} a^{0q}(|z|)\bar{z}^q.$$

Then by the similar argument as in the proof of Theorem 3.2, we can easily deduce that $e^{\frac{1}{4}|z|^2} f_m \times \varphi_k^0$ is a polynomial. In fact these are the only functions for which $e^{\frac{1}{4}|z|^2} f \times \varphi_k^0$ is a polynomial. This can be seen from Equation (3.5), when we reverse the process using the Hecke-Bochner identities. Thus the

space considered is just not empty, it includes all the sequence (f_m) which converges to

$$f(z)=\sum_{p=0}^{\infty}a^{p0}(|z|)z^p+\sum_{q=0}^{\infty}a^{0q}(|z|)\bar{z}^q$$

in $L^2(\mathbb{C})$. We believe that the proof of Theorem 3.4, without exponential condition on spectral projection would need a finer argument and hence we prefer to return to this question later.

Next, we prove the stronger result that X-axis together with Y-axis is a set of injectivity for the TSM for any function in $L^q(\mathbb{C})$.

For $\eta\in\mathbb{C}$, define the left twisted translate by

$$\tau_\eta f(\xi)=f(\xi-\eta)e^{\frac{i}{2}\mathrm{Im}(\eta.\bar{\xi})}.$$

Then $\tau_\eta(f\times\mu_r)=\tau_\eta f\times\mu_r$. Let S be a set of injectivity for the TSM on $L^q(\mathbb{C})$. Suppose $f\times\mu_r(z-\eta)=0,\forall r>0$ and $\forall z\in S$. Then

$$\tau_\eta f\times\mu_r(z)=\tau_\eta(f\times\mu_r)(z)=e^{\frac{i}{2}\mathrm{Im}(\eta.\bar{z})}f\times\mu_r(z-\eta)=0,$$

for all $r>0$ and $\forall z\in S$. Since the space $L^q(\mathbb{C})$ is twisted translations invariant, it follows that a set $S\subset\mathbb{C}$ is set of injectivity for the TSM if and only if for each $\eta\in\mathbb{C}$, the set $S-\eta$ is a set of injectivity for the TSM. That is, the Euclidean translate of the set S is also a set of injectivity for the TSM on $L^q(\mathbb{C})$. By rotation and translation, it is obvious that any two perpendicular lines can be set of injectivity for the TSM, provided X-axis and Y-axis together is a set of injectivity for the TSM.

Theorem 3.6. *Let $f\in L^q(\mathbb{C})$, for $1\leq q\leq 2$. Suppose $f\times\mu_r(x)=f\times\mu_r(ix)=0,\forall\ r>0$ and $\forall\ x\in\mathbb{R}$. Then $f=0$ a.e.*

Let $f\in L^q(\mathbb{C})$. Then by convolving f with a right and radial compactly supported smooth approximate identity, we can assume $f\in L^2(\mathbb{C})$. Let us consider the Hecke-Bochner-Laguerre series

$$Q_k(z)=\sum_{p=1}^{k}\left(C_{k-p}^{p0}z^p\varphi_{k-p}^p(z)+C_k^{0p}\bar{z}^p\varphi_k^p(z)\right)+\sum_{p=k+1}^{\infty}C_k^{0p}\bar{z}^p\varphi_k^p(z). \tag{3.6}$$

The proof is now based on symmetries and then cancelations. We decompose the above series into four (disjoint) series, each of which after equating its coefficients to zero, gives a system of solvable recursion relations. Using these recursion relations together with some basic properties of Laguerre polynomials, we show that all the coefficients appeared in the series (3.6) are zero.

Let $\mathbb{E}_+$ and $\mathbb{O}_+$ denote the sets of even and odd positive integers respectively. Let $E_k=\mathbb{E}_+\cap\{1,2,\ldots,k\}$, $F_k=\mathbb{E}_+\cap\{k+1,k+2,\ldots\}$, $G_k=\mathbb{O}_+\cap\{1,2,\ldots,k\}$ and $H_k=\mathbb{O}_+\cap\{k+1,k+2,\ldots\}$. Then, we can decompose the above series as $Q_k(z)=U_k(z)+V_k(z^2)$, where

$$U_k(z)=\sum_{p\in G_k}\left(C_{k-p}^{p0}z^p\varphi_{k-p}^p(z)+C_k^{0p}\bar{z}^p\varphi_k^p(z)\right)+\sum_{p\in H_k}C_k^{0p}\bar{z}^p\varphi_k^p$$

and
$$V_k(z^2) = \sum_{p\in E_k} \left(C_{k-p}^{p0} z^p \varphi_{k-p}^p(z) + C_k^{0p} \bar{z}^p \varphi_k^p(z) \right) + \sum_{p\in F_k} C_k^{0p} \bar{z}^p \varphi_k^p.$$
We shall call U_k and V_k as odd and even series respectively. For $x \in \mathbb{R}$, it is given that $U_k(x) + V_k(x^2) = 0$. Using the argument $x \to -x$, we have $U_k(x) = V_k(x^2) = 0$. Similarly, $U_k(ix) + V_k((ix)^2) = 0$, implies $U_k(ix) = V_k((ix)^2) = 0$. Indeed, we can put these conditions as follow.

(A): $U_k(x) = U_k(ix) = 0$ and

(B): $V_k(x^2) = V_k((ix)^2) = 0$.

In order to prove Theorem 3.6, we prove that $Q_k \equiv 0, \forall k \geq 0$. Now, we divide the proof into two parts: $0 \leq k \leq 3$ and $k \geq 4$.

Lemma 3.7. *Let $f \in L^2(\mathbb{C})$ and $0 \leq k \leq 3$. Suppose $Q_k(x) = Q_k(ix) = 0, \ \forall x \in \mathbb{R}$. Then $Q_k \equiv 0$.*

Proof. Since, we have shown in Theorem 3.2 that $Q_0 \equiv Q_1 \equiv 0$, we only need to prove $Q_k \equiv 0$ for $k = 2, 3$. For $k = 2$, by conditions (A), we get a pair of equations
$$x \left(C_1^{10} \varphi_1^1(x) + C_2^{01} \varphi_2^1(x) \right) + \sum_{m=2}^{\infty} C_2^{0,2m-1} x^{2m-1} \varphi_2^{2m-1}(x) = 0,$$
$$x \left(C_1^{10} \varphi_1^1(x) - C_2^{01} \varphi_2^1(x) \right) + \sum_{m=2}^{\infty} (-1)^m C_2^{0,2m-1} x^{2m-1} \varphi_2^{2m-1}(x) = 0.$$
On adding these two equations, we have
$$x C_1^{10} \varphi_1^1(x) + \sum_{m=2}^{\infty} C_2^{0,4m-5} x^{4m-5} \varphi_2^{4m-5}(x) = 0,$$
By equating the coefficients of $x, x^3, x^7, \ldots$ to zero, we get $C_1^{10} = 0$ and $C_2^{0,4m-5} = 0$, for $m = 2, 3, \ldots$. Similarly by subtracting and then equating the coefficients of $x, x^5, x^9, \ldots$ to zero, we obtain $C_2^{01} = 0$ and $C_2^{0,4m-7} = 0$, for $m = 3, 4, \ldots$. Hence $U_2 \equiv 0$. By conditions (B), we have
$$x^2 \left(C_0^{20} \varphi_0^2 + C_2^{02} \varphi_2^2 \right) + \sum_{m=2}^{\infty} C_2^{0,2m} x^{2m} \varphi_2^{2m}(x) = 0,$$
$$-x^2 \left(C_0^{20} \varphi_0^2 + C_2^{02} \varphi_2^2 \right) + \sum_{m=2}^{\infty} (-1)^m C_2^{0,2m} x^{2m} \varphi_2^{2m}(x) = 0.$$
In a quite similar way, we find $V_2 \equiv 0$ and hence $Q_2 \equiv 0$. Here, by adding and subtracting, we get the coefficients to be more disjoint, which is the only difficulty we need to resolve. The method used in this case will be repeated for $k \geq 3$.

For $k = 3$, by conditions (A), we have
$$x \left(C_2^{10} \varphi_2^1(x) + C_3^{01} \varphi_3^1(x) \right) + \sum_{m=3}^{\infty} C_3^{0,2m-1} x^{2m-1} \varphi_3^{2m-1}(x) = 0,$$

$$x\left(C_2^{10}\varphi_2^1(x)-C_3^{01}\varphi_3^1(x)\right)+\sum_{m=3}^{\infty}(-1)^m C_3^{0,2m-1}x^{2m-1}\varphi_3^{2m-1}(x)=0.$$

As very similar to above, the pair of equations implies that $U_3\equiv 0$. By conditions (B), we have

$$x^2\left(C_1^{20}\varphi_1^2(x)+C_3^{02}\varphi_3^2(x)\right)+\sum_{m=2}^{\infty}C_3^{0,2m}x^{2m}\varphi_3^{2m}(x)=0,$$

$$-x^2\left(C_1^{20}\varphi_1^2(x)+C_3^{02}\varphi_3^2(x)\right)+\sum_{m=2}^{\infty}(-1)^m C_3^{0,2m}x^{2m}\varphi_3^{2m}(x)=0.$$

This shows that $V_3\equiv 0$ and hence $Q_3\equiv 0$. $\square$

Next, we prove the following lemma for the case $k\geq 4$, which completes the proof of Theorem 3.6.

Lemma 3.8. *Let $f\in L^2(\mathbb{C})$ and $k\geq 4$. Suppose $Q_k(x)=Q_k(ix)=0,\ \forall x\in\mathbb{R}$. Then $Q_k\equiv 0$.*

Proof. First, we show that the odd series $U_k\equiv 0,\forall k\geq 4$. By conditions (A), we can write

$$\sum_{p\in G_k}x^p\left(C_{k-p}^{p0}\varphi_{k-p}^p(x)+C_k^{0p}\varphi_k^p(x)\right)+\sum_{p\in H_k}C_k^{0p}x^p\varphi_k^p(x)=0$$

$$\sum_{p\in G_k}(-1)^{\frac{p-1}{2}}x^p\left(C_{k-p}^{p0}\varphi_{k-p}^p(x)-C_k^{0p}\varphi_k^p(x)\right)+\sum_{p\in H_k}(-1)^{\frac{p+1}{2}}C_k^{0p}x^p\varphi_k^p(x)=0.$$

By adding and subtracting, we get two series in which there are no common coefficients. On equating the coefficients of $x,x^3,x^5,\ldots,$ in both the new series to zero, we get all the coefficients of odd series U_k are zero. Hence $U_k\equiv 0,\forall k\geq 4$. Since $i^4=1$, it shows that there must occur some change in the pattern of the even series V_k for $k\geq 4$. For instance consider V_4. By condition (B), we have

$$x^2\left(C_2^{20}\varphi_2^2(x)+C_4^{02}\varphi_4^2(x)\right)+x^4\left(C_0^{40}\varphi_0^4(x)+C_4^{04}\varphi_4^4(x)\right)+\sum_{m=3}^{\infty}C_4^{0,2m}x^{2m}\varphi_4^{2m}(x)=0,$$

$$-x^2\left(C_2^{20}\varphi_2^2(x)+C_4^{02}\varphi_4^2(x)\right)+x^4\left(C_0^{40}\varphi_0^4(x)+C_4^{04}\varphi_4^4(x)\right)+\sum_{m=3}^{\infty}(-1)^m C_4^{0,2m}x^{2m}\varphi_4^{2m}(x)=0.$$

On adding and subtracting, we get the two series

$$x^4\left(C_0^{40}\varphi_0^4(x)+C_4^{04}\varphi_4^4(x)\right)+\sum_{m=4}^{\infty}C_4^{0,4(m-2)}x^{2m}\varphi_4^{4(m-2)}(x)=0, \tag{3.7}$$

$$x^2\left(C_2^{20}\varphi_2^2(x)+C_4^{02}\varphi_4^2(x)\right)+\sum_{m=3}^{\infty}C_4^{0,4m-6}x^{4m-6}\varphi_4^{4m-6}(x)=0. \tag{3.8}$$

By equating the coefficients of $x^4, x^6, x^8, \ldots,$ in Equation (3.7) to zero, we get, all the coefficients in (3.7) are zero. By canceling $e^{-\frac{1}{4}x^2}$ in series (3.8) and using $x^2 \to 2x$, we have

$$2x\left(C_2^{20}L_2^2(x) + C_4^{02}L_4^2(x)\right) + \sum_{m=3}^{\infty} C_4^{0,4m-6}(2x)^{2m-3}L_4^{4m-6}(x) = 0.$$

On equating the coefficients of x and x^2 to zero, we get

$$\begin{pmatrix} L_2^2(0) & L_4^2(0) \\ (L_2^2)'(0) & (L_4^2)'(0) \end{pmatrix}\begin{pmatrix} C_2^{20} \\ C_4^{02} \end{pmatrix} = \begin{pmatrix} 6 & 6 \\ -4 & -20 \end{pmatrix}\begin{pmatrix} C_2^{20} \\ C_4^{02} \end{pmatrix} = 0. \tag{3.9}$$

Thus $C_2^{20} = C_4^{02} = 0$ and hence we find $V_4 \equiv 0$. Equivalently, we can use onwards to write: on equating the coefficients of x^2 and x^4 in Equation (3.8) to zero, we get Equations (3.9). Now, it only remains to show that $V_k \equiv 0, \forall k \geq 5$. In this case, by conditions (B), we have

$$\sum_{p\in E_k} x^p\left(C_{k-p}^{p0}\varphi_{k-p}^p + C_k^{0p}\varphi_k^p\right) + \sum_{p\in F_k} C_k^{0p}x^p\varphi_k^p = 0,$$

$$\sum_{p\in E_k} (-1)^{\frac{p}{2}}x^p\left(C_{k-p}^{p0}\varphi_{k-p}^p + C_k^{0p}\varphi_k^p\right) + \sum_{p\in F_k} (-1)^{\frac{p}{2}}C_k^{0p}x^p\varphi_k^p = 0.$$

We further require a partition of the set $\mathbb{E} = A_1 \cup A_2$, where $A_1 = \{4t-2 : t \in \mathbb{N}\}$ and $A_2 = \{4t : t \in \mathbb{N}\}$. By adding and subtracting, we will get the following pair of series having brackets.

$$\sum_{p\in E_k\cap A_2} x^p\left(C_{k-p}^{p0}\varphi_{k-p}^p + C_k^{0p}\varphi_k^p\right) + \sum_{p\in F_k\cap A_2} C_k^{0p}x^p\varphi_k^p = 0, \tag{3.10}$$

$$\sum_{p\in E_k\cap A_1} x^p\left(C_{k-p}^{p0}\varphi_{k-p}^p + C_k^{0p}\varphi_k^p\right) + \sum_{p\in F_k\cap A_1} C_k^{0p}x^p\varphi_k^p = 0, \tag{3.11}$$

Since the matrix

$$\begin{pmatrix} L_{k-p}^p(0) & L_k^p(0) \\ (L_{k-p}^p)'(0) & (L_k^p)'(0) \end{pmatrix}$$

is non-singular and $p \in A_2$, the brackets in (3.10) is not a problem. Hence on equating the coefficients of $x^2, x^4, x^6, \ldots$ to zero in (3.10), it follows that all the coefficients in (3.10) are zero. Similarly, in (3.11), as p in A_1, it also follows that all the coefficients in (3.11) are zero. Thus we find $V_k \equiv 0, \forall k \geq 5$ and hence $Q_k \equiv 0, \forall k \geq 0$. This completes the proof. □

Remark 3.9. (a). We can also prove the general case by calculating case-wise, when $k = 2m, 2m+1$ and $p = 4t-2, 4t$, but it would only make the calculation to be more complicated. In another attempt, to get a more transparent proof of Theorem 3.6, keep applying the right invariant operator $\tilde{A} = \frac{\partial}{\partial z} + \frac{1}{4}\bar{z}$ to $Q_k(z)$. Then a straightforward calculation shows that

$$\tilde{A}^p Q_k(0) = p!\varphi_{k-p}^p(0)C_{k-p}^{p0},$$

if $p \leq k$ and 0 otherwise. By the condition $Q_k(x) = Q_k(ix) = 0, \forall x \in \mathbb{R}$, it follows that $\tilde{A}Q_k(0) = 0$. This implies $C_{k-1}^{10} = 0$ and in turn $C_k^{01} = 0$. In view of this, we can immediately conclude that $Q_1 \equiv 0$. But for $k \geq 2$, the conditions on Q_k do not imply $\tilde{A}^p Q_k(0) = 0$, when $p \geq 2$, otherwise this would leads to a more transparent proof of Theorem 3.6.

(b). In Theorem 3.6, we have shown that any two lines having angle $\pi/2$ is a set of injectivity for the TSM on $\mathbb{C}$. However, the question that any two lines having positive angle less than $\pi/2$ can be a set of injectivity for the TSM on $\mathbb{C}$ is still unanswered.

(c). Consider Coxeter system of N-lines $\Sigma_N = \cup_{l=0}^{N-1}\{te^{i\theta_l} : \theta_l = \frac{\pi l}{N}, t \in \mathbb{R}\}$. Suppose $\theta_l = \pi/2$. Then $l = N/2$. By Theorem 3.6, it follows that any Coxeter system of even number of lines is also a set of injectivity for the TSM on $L^q(\mathbb{C})$.

Next, we set to describe the problem for any Coxeter system of odd lines. Let $1, \omega, \omega^2, \ldots, \omega^{N-1}$ be the N roots of unity. Then $\Sigma_N = \cup_{l=0}^{N-1}\{\omega^l x : x \in \mathbb{R}\}$. We have formulated this problem in the following way.

Conjecture. Let $f \in L^2(\mathbb{C})$. Suppose $f \times \mu_r(\omega^l x) = 0, \forall r > 0$, and $l = 0, 1, \ldots, N-1$ and $\forall\, x \in \mathbb{R}$. Then $f = 0$ a.e.

We would like to produce some evidence about the feasibility of this problem. In this case, we also get a system of solvable recursion relations by decomposing the series using symmetries, however those recursion relations for higher values of k, gives rise to a higher order square matrix, which is needed to show non-singular. This is the only difficulty in getting a solution to this problem. Since $Q_k(\omega^l x) = 0, \forall l;\ l = 0, 1, \ldots, N-1$ and $\forall x \in \mathbb{R}$, as similar to Theorem 3.6, we will have the following conditions.

(A): $U_k(\omega^l x) = 0, \forall\, l;\ l = 0, 1, \ldots, N-1$ and
(B): $V_k((\omega^l x)^2) = 0, \forall\, l;\ l = 0, 1, \ldots, N-1$.

For $N = 3$, by a simple argument that find all odd positive integers which are not divisible by 3, we can find a partition of the set of natural numbers as $\mathbb{N} = \cup_{i=0}^{2} A_i$, where $A_o = \{1, 2, 3, 4\}$, $A_1 = \{6t-1, 6t : t \in \mathbb{N}\}$ and $A_2 = \{6t+1, 6t+2, 6t+3, 6t+4 : t \in \mathbb{N}\}$.

For $k \in A_o$, the conditions (A) together with the facts $1 + \omega + \omega^2 = 0$ and $\omega^3 = 1$, implies that

$$C_k^{03} x^3 \varphi_k^3 + C_k^{09} x^9 \varphi_k^9 + \cdots = 0.$$

On equating the coefficient of $x^3, x^9, \ldots$ to zero, we get $C_k^{03} = C_k^{09} = \cdots = 0$. Hence the odd series U_k reduces to

$$x\left(C_{k-1}^{10}\varphi_{k-1}^1 + C_k^{01}\varphi_k^1\right) + C_k^{05} x^5 \varphi_k^5 + C_k^{07} x^7 \varphi_k^7 + \cdots = 0. \tag{3.12}$$

On equating the coefficients of x and x^3 in Equation (3.12) to zero, we get

$$\begin{pmatrix} L_{k-1}^1(0) & L_k^1(0) \\ (L_{k-1}^1)'(0) & (L_k^1)'(0) \end{pmatrix} \begin{pmatrix} C_{k-1}^{10} \\ C_k^{01} \end{pmatrix} = \begin{pmatrix} k & k+1 \\ \frac{-k(k-1)}{2} & \frac{-k}{2} \end{pmatrix} \begin{pmatrix} C_{k-1}^{10} \\ C_k^{01} \end{pmatrix} = 0. \tag{3.13}$$

Thus $C^{10}_{k-1} = C^{01}_{k} = 0$ and hence we find $U_k \equiv 0, \forall k \in A_o$. For $k \in \{1, 2, 3\}$, by condition (B) and the facts $1 + \omega + \omega^2 = 0$ and $w^3 = 1$, we have

$$C^{06}_{k} x^6 \varphi^6_k + C^{0,12}_{k} x^{12} \varphi^{12}_k + \cdots = 0.$$

This shows that $C^{06}_{2} = C^{0,12}_{2} = \cdots = 0$, and in turn the even series V_k reduces to

$$x^2 \left(C^{20}_{k-2} \varphi^{k}_{k-2} + C^{02}_{k} \varphi^2_k \right) + C^{04}_{k} x^4 \varphi^4_k + C^{08}_{k} x^8 \varphi^8_k + \cdots = 0.$$

Let $F^n_k = L^n_k(0)$. On equating the coefficients of x^2, x^4 and x^6 to zero, we get

$$\begin{pmatrix} F^2_{k-2} & F^2_k & 0 \\ F^2_0 & -F^3_{k-1} & F^4_k \\ 0 & F^4_{k-2} & -F^5_{k-1} \end{pmatrix} \begin{pmatrix} C^{20}_{k-2} \\ C^{02}_{k} \\ C^{04}_{k} \end{pmatrix} = 0.$$

This implies $C^{20}_{k-2} = C^{02}_{k} = C^{04}_{k} = 0$. Thus, it follows that $V_k \equiv 0$ and hence $Q_k \equiv 0$, for $k \in \{0, 1, 2, 3\}$. This gives a strong evidence about the existence of the Problem. We would also like to focus on to the proof, for higher values of k. Let $k \in A_1 \cup A_2$. Then by the conditions (A) together with the facts $1 + \omega + \omega^2 = 0$ and $\omega^3 = 1$, the odd Series U_k reduces to

$$\begin{aligned} & x \left(C^{10}_{k-1} \varphi^1_{k-1} + C^{01}_{k} \varphi^1_k \right) + x^5 \left(C^{50}_{k-5} \varphi^5_{k-5} + C^{05}_{k} \varphi^5_k \right) + x^7 \left(C^{70}_{k-7} \varphi^7_{k-7} + C^{07}_{k} \varphi^7_k \right) \\ & + \cdots + x^r \left(C^{r0}_{k-r} \varphi^r_{k-r} + C^{0r}_{k} \varphi^r_k \right) + \cdots + C^{0j}_{k} x^j \varphi^j_k + \cdots = 0. \end{aligned}$$

On equating the coefficients of x and x^3 to zero, we get $C^{10}_{k-1} = C^{01}_{k} = 0$. Thus

$$\begin{aligned} & x^5 \left(C^{50}_{k-5} \varphi^5_{k-5} + C^{05}_{k} \varphi^5_k \right) + x^7 \left(C^{70}_{k-7} \varphi^7_{k-7} + C^{07}_{k} \varphi^7_k \right) + \cdots \\ & + x^r \left(C^{r0}_{k-r} \varphi^r_{k-r} + C^{0r}_{k} \varphi^r_k \right) + \cdots + C^{0j}_{k} x^j \varphi^j_k + \cdots = 0. \end{aligned} \tag{3.14}$$

The main problem here is to remove the brackets (mixed term) in this series. Then, it is easy to show that rest of coefficients are zero. To remove the brackets, we need to identify the indices r, j, appeared in (3.14) and the number of equations m required. By Equation (3.14), we can write the following table.

k	r	j	$m = \frac{j+1}{2} - 2$	$k - r$
$k \in A_1$	$6t - 1$	$2k - 1$	$k - 2$	$0, 1$
$k \in A_2$	$6t + 1$	$2k + 1$	$k - 1$	$0, 1, 2, 3$

Let $k \in A_1$. Equate the coefficients of $x^5, x^7, \ldots, x^j$ to zero. In order to show that coefficients in the brackets are zero, we need to show that the following matrices are non-singular. For $k = 5$, we have the matrix

$$\begin{pmatrix} F^5_0 & F^5_5 & 0 \\ 0 & -F^6_4 & F^7_5 \\ 0 & F^7_3 & -F^8_4 \end{pmatrix},$$

which is non-singular. For higher values of k, we get the higher order matrices which should be non-singular. This is the only difficulty in the above arguments. Hence, we leave this problem open for future research.

Remark 3.10. In the case, when $N = 1$ (without exponential decay), we can not obtain the $m \times m$ matrices. However for $N \geq 3$, we have the $m \times m$ matrices which is needed to be non-singular.

From Remark 3.9(b), it is clear that the set Σ_{2N} is a set of injectivity for the TSM for $L^q(\mathbb{C})$, with $1 \leq q \leq 2$. As a dual problem, it is natural to ask that Σ_{2N} is a set of density for $L^p(\mathbb{C})$, for $2 \leq p < \infty$. The following result would emerge as the first result about the sets of density in terms of the TSM. Let $C_c^\sharp(\mathbb{C})$ denote the space of radial compactly supported continuous functions on $\mathbb{C}$. Let $\tau_z f(w) = f(z-w)e^{\frac{i}{2}\mathrm{Im}(z.\bar{w})}$.

Proposition 3.11. *The subspace* $\mathscr{F}(\Sigma_{2N}) = \mathrm{Span}\left\{\tau_z f : z \in \Sigma_{2N}, f \in C_c^\sharp(\mathbb{C})\right\}$ *is dense in* $L^p(\mathbb{C})$*, for* $2 \leq p < \infty$.

Proof. Let $\frac{1}{p} + \frac{1}{q} = 1$. Then $1 \leq q \leq 2$. By Hahn-Banach theorem, it is enough to show that $\mathscr{F}(\Sigma_{2N})^\perp = \{0\}$. Let $g \in L^q(\mathbb{C})$ be such that

$$\int_{\mathbb{C}} \tau_z f(w) g(w) dw = 0, z \in \Sigma_{2N},\ \forall f \in C_c^\sharp(\mathbb{C}).$$

That is,

$$\overline{\bar{g} \times \bar{f}}(z) = f \times g(z) = 0.$$

Let the support of f be contained in $[0, t]$. Then by passing to the polar decomposition, we get

$$\int_{r=0}^{t} \bar{g} \times \mu_r(z) \bar{f}(r) r^{2n-1} dr = 0.$$

By differentiating the above equation, it follows that $\bar{g} \times \mu_t(z) = 0, \forall t > 0$ and $\forall z \in \Sigma_{2N}$. Thus by Theorem 3.6, we conclude that $g = 0$ a.e. on $\mathbb{C}^n$. $\square$

4. Discussion on sets of injectivity in higher dimension

More generally, similar to the work of Agranovsky and Quinto [2], let $f \in L^1_{\text{loc}}(\mathbb{C}^n)$ and write $S(f) = \{z \in \mathbb{C}^n : f \times \mu_r(z) = 0, \forall\ r > 0\}$. Our main problem is to describe the complete geometrical structure of $S(f)$ that would ensure which "sets" are sets of injectivity for the TSM. There is one such result.

Lemma 4.1. *Let* $f \in L^p(\mathbb{C}^n) \cap C(\mathbb{C}^n)$*, for* $1 \leq p \leq \infty$*. Then*

$$S(f) = \bigcap_{k=0}^{\infty} Q_k^{-1}(0).$$

Proof. Let $z \in S(f)$. Then by polar decomposition, it follows that $Q_k(z) = 0, \forall k \geq 0$. Conversely, let $Q_k(z) = 0, \forall k \geq 0$. Then

$$\int_{r=0}^{\infty} f \times \mu_r(z) \varphi_k^{n-1}(r) r^{2n-1} dr = 0, \forall k \geq 0.$$

Since the set $\{\varphi_k^{n-1} : k = 0, 1, 2, \ldots\}$ is an orthonormal set for $L^2\left(\mathbb{R}_+, r^{2n-1}dr\right)$ and $f \times \mu_r(z)$ is continuous in r, it follows that $f \times \mu_r(z) = 0, \forall r > 0$ and hence $z \in S(f)$. □

Next, we find out $S(f)$ for the type function $f(z) = \tilde{a}(|z|)P(z)$ on $\mathbb{C}^n$. For this, we need the following result of Filaseta and Lam [9], about the irreducibility of Laguerre polynomials. Define the Laguerre polynomials by

$$L_k^\alpha(x) = \sum_{i=0}^{k} (-1)^i \binom{\alpha + k}{k - i} \frac{x^i}{i!},$$

where $k \in \mathbb{Z}_+$ and $\alpha \in \mathbb{C}$.

Theorem 4.2. [9] *Let α be a rational number, which is not a negative integer. Then for all but finitely many $k \in \mathbb{Z}_+$, the polynomial $L_k^\alpha(x)$ is irreducible over the rationals.*

Using Theorem 4.2, we have obtained the following corollary about the zeros of Laguerre polynomials.

Corollary 4.3. *Let $k \in \mathbb{Z}_+$. Then for all but finitely many k, the Laguerre polynomials $L_k^{n-1}(x)$'s have distinct zeros over the reals.*

Proof. By Theorem 4.2, there exists $k_o \in \mathbb{Z}_+$ such that L_k^{n-1}'s are irreducible over $\mathbb{Q}$ whenever $k \geq k_o$. Therefore, we can find polynomials $P_1, P_2 \in \mathbb{Q}[x]$ such that $P_1 L_{k_1}^{n-1} + P_2 L_{k_2}^{n-1} = 1$, over $\mathbb{Q}$ with $k_1, k_2 \geq k_o$. Since this identity continue to hold on $\mathbb{R}$, it follows that $L_{k_1}^{n-1}$ and $L_{k_2}^{n-1}$ have no common zero over $\mathbb{R}$. □

Proposition 4.4. *Let f be a non-zero type function $f = \tilde{a}P \in L^2(\mathbb{C}^n)$, where $P \in H_{p,q}$. Then $S(f) = P^{-1}(0) \cup F$, where F is a finite union of spheres in $\mathbb{C}^n$.*

Proof. Since $f \not\equiv 0$, there exists at least some $k \in \mathbb{Z}_+$ for which $Q_k^{-1}(0) \neq \mathbb{C}^n$. Therefore,

$$Q_k^{-1}(0) = P^{-1}(0) \cup \left(\varphi_{k-p}^{n+p+q-1}\right)^{-1}(0),$$

for some $k \in \mathbb{Z}_+$. Hence $S(f) = P^{-1}(0) \cup F$. □

Proposition 4.5. *Let $f = \tilde{a}P \in L^2(\mathbb{C}^n)$, where $P \in H_{p,q}$. Suppose Q_k is not identically zero on $\mathbb{C}^n$, for all but finitely many k. Then $S(f) = P^{-1}(0)$.*

Proof. Since $Q_k(z) = f \times \varphi_k^{n-1}$. Then by Lemma 4.1, we have $S(f) = \cap_{k=0}^{\infty} Q_k^{-1}(0)$. By Hecke-Bochner identity, we can write

$$Q_k(z) = (2\pi)^n \left\langle \tilde{a}, \varphi_{k-p}^{n+p+q-1} \right\rangle P(z) \varphi_{k-p}^{n+p+q-1}(z).$$

Since $Q_k \not\equiv 0$ for infinitely many $k \in \mathbb{Z}_+$. Therefore,

$$Q_k^{-1}(0) = P^{-1}(0) \cup \left(\varphi_{k-p}^{n+p+q-1}\right)^{-1}(0) \neq \mathbb{C}^n,$$

for infinitely many k. In view of Corollary 4.3, the functions $\varphi_{k-p}^{n+p+q-1}$'s can not have a common zero except for finitely many $k \in \mathbb{Z}_+$ with $k \geq p$. Hence, we conclude that $S(f) = P^{-1}(0)$. □

Now, we would like to address the problem in the higher dimensional space $\mathbb{C}^n$ with $n \geq 2$. Let $f \in L^2(\mathbb{C}^n)$. Consider the spherical harmonic decomposition of f as

$$f(z) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{j=1}^{d(p,q)} \tilde{a}_j^{pq}(|z|)P_{pq}^j(z). \tag{4.1}$$

In view of the Hecke-Bochner identities (2.3), we conclude that

$$\begin{aligned} f \times \varphi_{k_0}^{n-1} &= \sum_{p=0}^{k_0}\sum_{q=0}^{\infty}\sum_{j=1}^{d(p,q)} C_{k_0-p,j}^{pq} P_{pq}^j \varphi_{k_0-p}^{n+p+q-1} \\ &= \sum_{p=0}^{k_0}\sum_{q=0}^{\infty} P_{pq}^{k_0}\varphi_{k_0-p}^{n+p+q-1}, \text{ where } P_{pq}^{k_0} \in H_{p,q}. \end{aligned}$$

Now look at the following concrete expression for the spectral projections

$$Q_k(z) = \sum_{p=0}^{k}\sum_{q=0}^{\infty} P_{pq}^k(z)\varphi_{k-p}^{n+p+q-1}(z), \;\; P_{pq}^k \in H_{p,q}. \tag{4.2}$$

Remark 4.6. (a). As very much similar to complex plane $\mathbb{C}$, our believe suggest that any Coxeter system of hyperplanes can be a set of injectivity for the TSM on $\mathbb{C}^n$. For instance on $\mathbb{C}^2$, suppose the function

$$Q(z_1, z_2) = \sum_{p=1}^{2}(a_p z_1^p + b_p z_2^p) + \sum_{q=1}^{2}(c_q \bar{z}_1{}^q + d_q \bar{z}_2{}^q)$$

vanishes on each of co-ordinate axis, i.e., $Q(x,0) = Q(ix,0) = Q(0,x) = Q(0,ix) = 0, \forall x \in \mathbb{R}$. Then $Q \equiv 0$. As another example, consider a typical polynomial $P(z_1, z_2) = cz_1^p \bar{z}_2^q \in H_{p,q}$. Suppose $P(z_1, x_2) = 0, \forall z_1 \in \mathbb{C}$ and $\forall x_2 \in \mathbb{R}$. Then $P \equiv 0$. In view of these arguments, write

$$S = (\mathbb{C}\times\mathbb{R}) \cup (\mathbb{C}\times i\mathbb{R}) \cup (\mathbb{R}\times\mathbb{C}) \cup (i\mathbb{R}\times\mathbb{C}).$$

It is natural to ask, whether the set S can be a set of injectivity for the TSM on $L^q(\mathbb{C}^2)$, for $1 \leq q \leq 2$.

(b). Let us rewrite $Q_k f = f \times \varphi_k^{n-1}$. From the explicit expression of Q_k, given by (4.2), it follows that

$$\|Q_k f\|_2^2 = \sum_{p=0}^{k}\sum_{q=0}^{\infty} \left\|Y_{pq}^k(f)\right\|_2^2 \left\|\varphi_{k-p}^{n+p+q-1}\right\|_2^2, \tag{4.3}$$

where $Y_{pq}^k(f)$'s are spherical harmonics depending upon f. In the work [22], Stempak and Zienkiewic have established that for $f \in L^r(\mathbb{C}^n)$, with $1 \leq r < \frac{2(2n+1)}{2n+3}$, the operators Q_k satisfy the estimate $\|Q_k f\|_2 \leq C_k\|f\|_r$. On the basis of the equality (4.3), it is natural to ask, whether the map $f \to f \times \varphi_k^{n-1}$ would satisfy an end point estimate.

(c). Let μ be a finite Borel measure which is supported on a curve γ and S be a non-empty set in $\mathbb{R}^2$. Then the pair (γ, S) is called a Heisenberg Uniqueness pairs (HUP) for μ if its Fourier transform $\hat{\mu}(x,y) = 0,\ \forall\ (x,y) \in S$, implies $\mu = 0$. In a recent work [12], Hedenmalm et al. prove the following result. Suppose μ is supported on the hyperbola $\gamma = \{(x,y) :\ \ xy = 1\}$ and $\hat{\mu}$ vanishes on the lattice-cross $S = \alpha\mathbb{Z} \times \{0\} \cup \{0\} \times \beta\mathbb{Z}$. Then $\mu = 0$ if and only if $\alpha\beta \leq 1$, where $\alpha, \beta \in \mathbb{R}_+$. This is a variance of uncertainty principle for Fourier transform. In view of this and the fact that $\varphi_k^0 \times \mu$ is real analytic, it is natural to ask the following question. Let μ be a finite measure supported on a real analytic curve γ and S be a non-empty set in $\mathbb{C}$. Then find all those non-trivial pair (γ, S) such that $\varphi_k^0 \times \mu(z) = 0,\ \forall z \in S$ and $\forall k \geq 0$. Implies $\mu = 0$. Here, we skip to write further details about these ideas and they might be appear in the successive work.

Concluding remarks: We would like to point out the key motivation behind X-axis together with Y-axis is a set of injectivity for the TSM on $\mathbb{C}$. Consider the function $Q(z) = c_0 z + c_1 \bar{z} + c_2 z^3 + c_3 \bar{z}^3$. Suppose $Q(x) = Q(ix) = 0, \forall x \in \mathbb{R}$. Then $Q \equiv 0$. This result can also be interpreted on the Heisenberg group for the spherical means given by (2.1). The set $\tilde{\Sigma}_{2N} = \cup_{l=0}^{2N-1}\{(\omega^l x, t) : x, t \in \mathbb{R}\}$ is a set of injectivity for the spherical means on $\mathbb{H}^1$.

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School of Mathematics, Harish-Chandra Research Institute, Allahabad, India 211019.

E-mail address: `rksri@hri.res.in`